General Relativity: $\boxed{\text{Week 6}}$

Last time: We introduced a hierarchy of causality conditions on a spacetime (M, g).

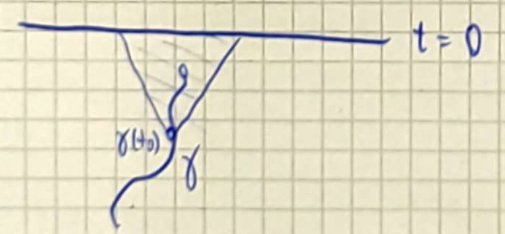
- Causal: No closed causal curves
- Strongly causal: No <u>almost</u> closed causal curves.
- Stably causal: $\exists$ time function $f: M \to \mathbb{R}$
- Globally hyperbolic: Admits a <u>Cauchy hypersurface</u> Σ

<u>Example</u>. Minkowski spacetime is globally hyperbolic.

Choose $\Sigma = \{t = 0\}$. Then.

- Since the function $t: \mathbb{R}^{n+1} \to \mathbb{R}$ is a time function (strictly monotonic along causal curves), every causal curve intersects Σ at most once.

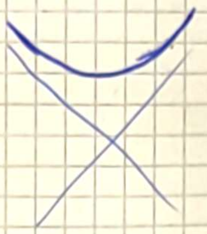
- Let γ be a future directed causal curve, $\gamma: (a,b) \to \mathbb{R}^{n+1}$. Assume it is inextendible. For the sake of contradiction, assume it doesn't intersect Σ. (Without loss of generality, assume that $\gamma \subset \{t < 0\}$.



Let $t_0 \in (a,b)$. Then $J^+(\gamma(t_0)) \cap \underbrace{\{t \leq 0\}}_{= J^-(\{t=0\})}$ is bounded. So $\gamma|_{[t_0, b)}$ is contained in a compact set of $\mathbb{R}^{n+1}$

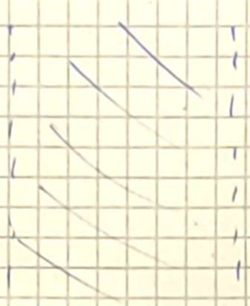
$\Rightarrow \exists\, t_n \to b$ s.t $\gamma(t_n) \longrightarrow p$. Since γ is a Lipschitz curve (being causal in Minkowski) $\Rightarrow$ $\gamma(t) \xrightarrow{t \to b^-} p$. So γ can be extended beyond $t = b \Rightarrow$ Contradiction.

• Note: The hyperboloid



$\{t^2 - \sum_{i=1}^{n} (x^i)^2 = 1\}$ is <u>not</u> a Cauchy hypersurface of $(\mathbb{R}^{n+1}, \eta)$.

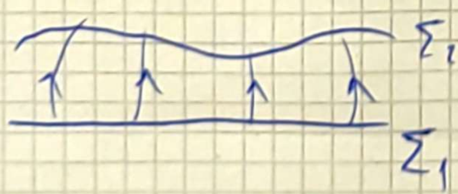
• The strip



$\subseteq (\mathbb{R}^{1+1}, \eta)$ is <u>not</u> globally hyperbolic.

<u>Lemma</u>: If Σ_1, Σ_2 are Cauchy hypersurfaces of (M, g) then they are homeomorphic.

<u>Proof</u>: Fix a timelike vector V (possible since (M,g) is a spacetime). Each flow line of V intersects Σ_1, Σ_2 exactly once, hence it defines a 1-1 and onto map $\Sigma_1 \longrightarrow \Sigma_2$. This map (and its inverse) is continuous, in view of the continuous dependence of the integral curves on their initial point.



$\boxed{}$

<u>Prop</u>: If (M, g) is globally hyperbolic and Σ is a Cauchy hypersurface, then $M \cong \mathbb{R} \times \Sigma$

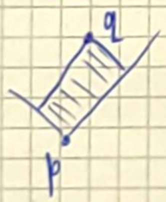
Sketch of the proof: Fix a timelike vector V on M, and "suitably normalize" it so that each integral curve exists for infinite time. Then the flow map $\phi^{(V)}: \mathbb{R} \times \Sigma \longrightarrow M$ provides the identification.

Equivalent definitions: (For a proof, see Wald Ch. 8 or Hawking-Ellis)

Let (M, g) be a spacetime:

Globally hyperbolic $\iff$ Strongly causal & $J^+(p) \cap J^-(q)$ is compact $\forall p, q$ $\iff$ Strongly causal & the space $C(p,q)$ of causal curves from p to q is compact in the C^0 topology

"Causal diamond" → $J^+(p) \cap J^-(q)$



Original definition by Leray ↗ ($C(p,q)$ compact)

Note: In Minkowski, if $\gamma \in C(p,q)$, then it is a Lipschitz curve when parametrized by t $\Rightarrow$ Compactness follows by Arzela-Ascoli

Global hyperbolicity: In many cases, it is the Lorentzian analogue of completeness.

E.g. If (M, g) is globally hyperbolic and $q \in J^+(p)$: Then there exists a causal geodesic connecting p to q (Follows from the compactness of $C(p,q)$ by maximizing the length).

And: Prop: If (M,g) is globally hyperbolic: $\exists$ smooth time function $t: M \longrightarrow \mathbb{R}$ such that, $\forall \tau \in \mathbb{R}$, $\Sigma_\tau = \{t = \tau\}$ is a Cauchy hypersurface.

(So in particular: Global hyperbolicity $\Rightarrow$ stable causality).

Riemann curvature tensor

Let (M,g) be a Lorentzian manifold. Riemann curvature tensor: The $(1,3)$ tensor field defined by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

It is a tensor field! (Check that it's C^∞-multilinear)

Components: $R^\alpha{}_{\beta\gamma\delta} = \left(R(\partial_\gamma, \partial_\delta)\partial_\beta\right)^\alpha$

We will frequently lower the index: $R_{\alpha\beta\gamma\delta} = g_{\alpha\lambda} R^\lambda{}_{\beta\gamma\delta}$

(Note: $R_{\alpha\beta\gamma\delta} = g(R(\partial_\gamma, \partial_\delta)\partial_\beta, \partial_\alpha) = g(R(\partial_\alpha, \partial_\beta)\partial_\delta, \partial_\gamma)$).

Note: In coordinates:

$$R^\alpha{}_{\beta\gamma\delta} = \partial_\gamma \Gamma^\alpha_{\delta\beta} - \partial_\delta \Gamma^\alpha_{\gamma\beta} + \Gamma^\alpha_{\gamma\lambda}\Gamma^\lambda_{\delta\beta} - \Gamma^\alpha_{\delta\lambda}\Gamma^\lambda_{\gamma\beta}$$

and $$R_{\alpha\beta\gamma\delta} = \frac{1}{2}\left(\partial^2_{\beta\gamma} g_{\alpha\delta} - \partial^2_{\alpha\gamma} g_{\beta\delta} + \partial^2_{\alpha\delta} g_{\beta\gamma} - \partial^2_{\beta\delta} g_{\alpha\gamma}\right) + g \cdot \partial g \cdot \partial g$$

Note: In normal coordinates: $\partial g(0) = 0$, so

$$R_{\alpha\beta\gamma\delta}(0) = \frac{1}{2}\left(\partial^2_{\beta\gamma} g_{\alpha\delta}(0) - \partial^2_{\alpha\gamma} g_{\beta\delta}(0) + \partial^2_{\alpha\delta} g_{\beta\gamma}(0) - \partial^2_{\beta\delta} g_{\alpha\gamma}(0)\right)$$

Symmetries of the Riemann tensor:

- $R_{\alpha\beta\gamma\delta} = -R_{\beta\alpha\gamma\delta} = -R_{\alpha\beta\delta\gamma}$
- $R_{\alpha\beta\gamma\delta} = R_{\gamma\delta\alpha\beta}$
- 1st Bianchi identity: $R_{\alpha\beta\gamma\delta} + R_{\gamma\alpha\beta\delta} + R_{\beta\gamma\alpha\delta} = 0$
- 2nd Bianchi identity: $\nabla_\alpha R_{\beta\gamma\delta\varepsilon} + \nabla_\gamma R_{\alpha\beta\delta\varepsilon} + \nabla_\beta R_{\gamma\alpha\delta\varepsilon} = 0$

This $(\nabla R)_{\alpha\beta\gamma\delta\varepsilon}$

<u>Sketch of proofs:</u> These are tensorial identities, so it suffices to prove them in one coordinate system.

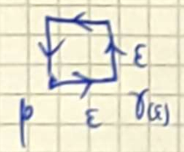
Choosing normal coordinates at p:

$$R_{\alpha\beta\gamma\delta}\big|_p = \frac{1}{2}\left(\partial^2_{\beta\gamma} g_{\alpha\delta} - \partial^2_{\alpha\gamma} g_{\beta\delta} + \partial^2_{\alpha\delta} g_{\beta\gamma} - \partial^2_{\beta\gamma} g_{\alpha\delta}\right)\Big|_p$$

and $\partial^2_{\alpha\beta} g_{\gamma\delta}\big|_p + \partial^2_{\gamma\alpha} g_{\beta\delta}\big|_p + \partial^2_{\beta\gamma} g_{\alpha\delta}\big|_p = 0.$ $\square$

Physical interpretation of the curvature tensor:

- When parallel transporting around a closed infinitesimal loop:



R measures the failure of $\mathbb{P}^{(\gamma_\varepsilon)}_{p\to p}$ to be the identity.

- <u>Tidal Forces:</u> When looking at nearby geodesics, R encodes the relevant acceleration

(see Jacobi fields eqn: $D_{\dot\gamma} D_{\dot\gamma} J + R(J,\dot\gamma)\dot\gamma = 0$)

On Minkowski: $R_{\alpha\beta\gamma\delta} = 0$ (Characterizes Minkowski !)

Ricci tensor: R_{ij} or $Ric_{ij} = R^{\alpha}{}_{i\alpha j} = g^{\alpha\beta} R_{\alpha i \beta j}$

(it is symmetric)

Ricci scalar (or scalar curvature): $R = g^{\alpha\beta} R_{\alpha\beta}$

Einstein tensor: $G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta}$

Using the Bianchi identities: We can show that G is always divergence free, i.e. $\nabla^{\alpha} G_{\alpha\beta} = 0$ (Exercise).

Weyl tensor: For (M^{n+1}, g):

$$W_{\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma\delta} - \frac{1}{n-1}\left(\left(Ric - \frac{R}{n+1} g\right) \owedge g\right)_{\alpha\beta\gamma\delta} - \frac{R}{2n(n+1)} \left(g \owedge g\right)_{\alpha\beta\gamma\delta},$$

Where $(h \owedge f)_{\alpha\beta\gamma\delta} = h_{\alpha\gamma} f_{\beta\delta} - h_{\beta\gamma} f_{\alpha\delta} + h_{\beta\delta} f_{\alpha\gamma} - h_{\alpha\delta} f_{\beta\gamma}$.

(Kulkarni - Nomizu product)

- It is the traceless part of R
- $W = 0$ always when $n \leq 2$.

Special relativity:

Formulated by Einstein in 1905. Geometric reformulation: Minkowski (1908)

Space and time: $(\mathbb{R}^{3+1}, \eta)$.

Inertial observers: Move in timelike straight lines